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Health Monitoring of in-service Bridge by Covariance of Covariance Matrix of acceleration responses

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ABSTRACT: A new matrix on the covariance of covariance is formed from the auto/cross-correlation function of acceleration responses of a structure under white noise ambient excitation. The components of the covariance matrix are proved to be function of the modal parameters (modal frequency, mode shape and modal damping) of the structure. The number of vibration modes associated with components of the matrix is only limited by the sampling frequency. Compared to the general methods for extracting modal parameters, the formulated covariance matrix contains more information on the vibration modes of the structure. An in-service suspension bridge is analyzed. Only the output acceleration responses are used to compute the covariance of covariance (CoC) matrix and the Pattern Assurance Criterion (PAC) values. From the CoCs variation and the changing trend of the PAC curve, the healthy condition of the bridge is assessed.

1 INTRODUCTION

The information required for system identification or damage detection of a dynamic system generally consists of both the input force and the resulting response. However, it would be very difficult and sometimes impossible to measure the actual excitation (such as wind, vehicular and wave excitation) for a large structure (such as bridges, offshore platforms, and wind turbines) [Shen et al (2003)]. The huge amount of energy necessary to create structural vibrations may cause local damage in the structure if it can be generated artificially. Therefore, system identification or damage detection is preferable to be done with the output response-only measurement.

A new covariance matrix is formed from the auto/cross-correlation function of the acceleration responses of a structure under ambient white noise excitation. The components of the covariance matrix are proved to be function of the structural modal parameters (modal frequency, mode shape and modal damping) of a structure. Information from all the vibration modes limited by the sampling frequency contributes to these components of the matrix. The method is applied on an in-service suspension bridge. Six accelerometers are installed vertically on the bridge and the acceleration responses are measured continuously. The available data from April 12, 2010 to June 6, 2010 are analyzed. Two hours' acceleration responses are used to compute the covariance of covariance matrix and its pattern assurance values. The trend of variation of the pattern assurance values is given and long-term health monitoring of the structure is performed.

2 COVARIANCE OF COVARIANCE (COC) MATRIX OF ACCELERATION RESPONSE

The equation of motion of a N degrees-of-freedom (DOFs) viscous damped structural system under support excitation is given as

$$\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{C}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=-\mathbf{M}\times\mathbf{L}\cdot\ddot{x}_s(t) \quad (1)$$

where $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ are the $N\times N$ mass, damping and stiffness matrices, respectively. Matrix $\mathbf{L}$ with a size of $N\times 1$ is the mapping vector relating the support DOFs with the corresponding DOFs of the system. $\mathbf{x}$, $\dot{\mathbf{x}}$, $\ddot{\mathbf{x}}$ are the $N\times 1$ displacement, velocity, acceleration vectors, respectively.

Environmental support excitation is universal for the whole structure, and its effect is represented in terms of $x_s, \dot{x}_s, \ddot{x}_s$, which are the displacement, velocity and acceleration, respectively, at the degrees-of-freedoms, s, of the support of the structure.

When $\ddot{x}_s$ is assumed to be of ideal white noise distribution, the autocorrelation function of $\ddot{x}_s$ is [Bendat & Piersol (1993)]

$$E(\ddot{x}_s(\sigma_1)\ddot{x}_s(\sigma_2))=S\delta(\sigma_1-\sigma_2) \quad (2)$$

where S is a constant defining the magnitude of excitation of $\ddot{x}_s$.
If $R_{pl}(\tau)$ denotes the cross-covariance of the accelerations from the pth and lth DOFs of the system, it can be written as follows [Li & Law (2010)],

$$R_{pl}(\tau)=S\sum_j\frac{\Phi_{lj}\phi_{fj}}{\omega_{dj}}e^{-\xi_j\omega_j\tau}\{G_j\cos(\omega_{dj}\tau)+H_j\sin(\omega_{dj}\tau)\} \quad (3)$$

where $G_j=\sum_i\frac{\Phi_{pi}\phi_{fi}}{\omega_{di}}A_{ij}$ and $H_j=\sum_i\frac{\Phi_{pi}\phi_{fi}}{\omega_{di}}B_{ij}$

$$\begin{aligned}A_{ij}=&\frac{1}{2}c_ic_j\left[\frac{s_i+s_j}{(s_i+s_j)^2+(\omega_{di}+\omega_{dj})^2}+\frac{s_i+s_j}{(s_i+s_j)^2+(\omega_{di}-\omega_{dj})^2}\right]\\&+\frac{1}{2}c_id_j\left[\frac{\omega_{di}+\omega_{dj}}{(s_i+s_j)^2+(\omega_{di}+\omega_{dj})^2}-\frac{\omega_{di}-\omega_{dj}}{(s_i+s_j)^2+(\omega_{di}-\omega_{dj})^2}\right]\\&+\frac{1}{2}d_ic_j\left[\frac{\omega_{di}+\omega_{dj}}{(s_i+s_j)^2+(\omega_{di}+\omega_{dj})^2}+\frac{\omega_{di}-\omega_{dj}}{(s_i+s_j)^2+(\omega_{di}-\omega_{dj})^2}\right]\\&-\frac{1}{2}d_id_j\left[\frac{s_i+s_j}{(s_i+s_j)^2+(\omega_{di}+\omega_{dj})^2}-\frac{s_i+s_j}{(s_i+s_j)^2+(\omega_{di}-\omega_{dj})^2}\right]\end{aligned} \quad (4)$$

and

$$\begin{aligned} B_{ij} = &-\frac{1}{2}c_i c_j \left[\frac{\omega_{di}+\omega_{dj}}{(s_i+s_j)^2+(\omega_{di}+\omega_{dj})^2} - \frac{\omega_{di}-\omega_{dj}}{(s_i+s_j)^2+(\omega_{di}-\omega_{dj})^2} \right] \\ &+\frac{1}{2}c_i d_j \left[\left[\frac{s_i+s_j}{(s_i+s_j)^2+(\omega_{di}+\omega_{dj})^2} + \frac{s_i+s_j}{(s_i+s_j)^2+(\omega_{di}-\omega_{dj})^2} \right] \right] \\ &+\frac{1}{2}d_i c_j \left[\left[\frac{s_i+s_j}{(s_i+s_j)^2+(\omega_{di}+\omega_{dj})^2} - \frac{s_i+s_j}{(s_i+s_j)^2+(\omega_{di}-\omega_{dj})^2} \right] \right] \\ &+\frac{1}{2}d_i d_j \left[\frac{\omega_{di}+\omega_{dj}}{(s_i+s_j)^2+(\omega_{di}+\omega_{dj})^2} + \frac{\omega_{di}-\omega_{dj}}{(s_i+s_j)^2+(\omega_{di}-\omega_{dj})^2} \right] \end{aligned} \tag{5}$$

where $c_i = -2\xi_i\omega_i\omega_{di}$, $d_i = \xi_i^2\omega_i^2 - \omega_{di}^2$, $s_i = \xi_i\omega_i$, Φ_{li} is the l th element of the i th mode shape Φ_i . ω_i , ω_{di} , ξ_i are the i th undamped modal frequency , damped modal frequency and damping ratio, and $\phi_{fi} = -\Phi_i^T \cdot \mathbf{M} \cdot \mathbf{L}$. The auto-covariance function can be derived in the same way by putting $l = p$ in Equation (3).

The covariance of acceleration response in Equation (3) can also be expressed as

$$R_{pl}(\tau) = \sum_j \Phi_{lj} u(p,\tau)_i = \mathbf{\Phi}_l \mathbf{u}(p,\tau) \tag{6}$$

where

$$u(p,\tau)_j = S\frac{\phi_{fj}}{\omega_{dj}} e^{-\xi_j\omega_j\tau} \{G_j \cos(\omega_{dj}\tau) + H_j \sin(\omega_{dj}\tau)\}$$

$$\mathbf{u}(p,\tau) = \left[u(p,\tau)_1 \quad u(p,\tau)_2 \quad \cdots \quad u(p,\tau)_N \right]^T .$$

With l referring to different DOFs, the CoC matrix (denoted by $\mathbf{T}_p$ in the rest of the paper) of acceleration responses is,

$$\mathbf{T}_p = \mathbf{R}_p \cdot \mathbf{R}_p^T = \mathbf{\Phi}(\mathbf{u}(p) \cdot (\mathbf{u}(p))^T)\mathbf{\Phi}^T \tag{7}$$

where

$$\mathbf{R}_p = \begin{bmatrix} R_{pl_1}(t_1) & R_{pl_1}(t_2) & \cdots & R_{pl_1}(t_{nt}) \\ R_{pl_2}(t_1) & R_{pl_2}(t_2) & \cdots & R_{pl_2}(t_{nt}) \\ \cdots & \cdots & \cdots & \cdots \\ R_{pl_m}(t_1) & R_{pl_m}(t_2) & \cdots & R_{pl_m}(t_{nt}) \end{bmatrix}, \ \mathbf{u}(p) = \begin{bmatrix} u(p,t_1)_1 & u(p,t_2)_1 & \cdots & u(p,t_{nt})_1 \\ u(p,t_1)_2 & u(p,t_2)_2 & \cdots & u(p,t_{nt})_2 \\ \cdots & \cdots & \cdots & \cdots \\ u(p,t_1)_N & u(p,t_2)_N & \cdots & u(p,t_{nt})_N \end{bmatrix}.$$

$\mathbf{l} = \left[l_1 \quad l_2 \quad \cdots \quad l_m \right]$ are the m DOFs from which the acceleration responses are measured, p is the reference DOF of the structure, $\mathbf{t} = \left[t_1 \quad t_2 \quad \cdots \quad t_{nt} \right]$ denotes the time instance of the covariance of the acceleration response. $\mathbf{T_p}$ is noted to be a symmetrical matrix.

The matrix $\mathbf{u}(p) \cdot (\mathbf{u}(p))^T$ in Equation (7) can be computed as

$$[\mathbf{u}(p)\cdot(\mathbf{u}(p))^T]_{i\times j}=\frac{S^2}{\Delta t}\frac{\phi_{fi}}{\omega_{di}}\frac{\phi_{fj}}{\omega_{dj}}\{G_iG_j\frac{1}{2}[\frac{s_i+s_j}{(s_i+s_j)^2+(\omega_{di}+\omega_{dj})^2}+\frac{s_i+s_j}{(s_i+s_j)^2+(\omega_{di}-\omega_{dj})^2}]$$
$$+G_iH_j\frac{1}{2}[\frac{\omega_{di}+\omega_{dj}}{(s_i+s_j)^2+(\omega_{di}+\omega_{dj})^2}-\frac{\omega_{di}-\omega_{dj}}{(s_i+s_j)^2+(\omega_{di}-\omega_{dj})^2}] \quad ,(8)$$
$$+H_iG_j\frac{1}{2}[\frac{\omega_{di}+\omega_{dj}}{(s_i+s_j)^2+(\omega_{di}+\omega_{dj})^2}+\frac{\omega_{di}-\omega_{dj}}{(s_i+s_j)^2+(\omega_{di}-\omega_{dj})^2}]$$
$$-H_iH_j\frac{1}{2}[\frac{s_i+s_j}{(s_i+s_j)^2+(\omega_{di}+\omega_{dj})^2}-\frac{s_i+s_j}{(s_i+s_j)^2+(\omega_{di}-\omega_{dj})^2}]\}$$

where the subscript $_{i\times j}$ denotes the element at i th row and j th column of the matrix $[\mathbf{u}(p)\cdot(\mathbf{u}(p))^T]$. Both i and j denote the mode number of the structure. Δt is the time step.

Equation (8) shows that matrix $\mathbf{u}(p)\cdot(\mathbf{u}(p))^T$ is only related to the mode shape, modal frequency and modal damping ratio of a structure. Then matrix $\mathbf{T_p}$ in Equation (7) is noted to be a function of the modal parameters of the structure. The change in the structural parameters is therefore related to the modal parameters and subsequently to the covariance of covariance matrix $\mathbf{T}_p$.

The above formulation is based on the case with the support excitation. If the ambient white noise excitation is applied on the structure instead of the support, similar formulations can be derived just by replacing $\phi_{fi}=-\Phi_i^T\cdot\mathbf{M}\cdot\mathbf{L}$ by $\phi_{fi}=\Phi_i^T\cdot\mathbf{L}$, where Matrix $\mathbf{L}$ is still the mapping vector relating the excitation DoFs to the corresponding DoFs of the structure.

3 MEASUREMENT ON IN-SERVICE STRUCTURES

Accelerometers are assumed to be installed on the monitored structure. The acceleration responses from selected measuring points of the structure under white noise excitation are recorded continuously. The covariance of the acceleration responses and the covariance of covariance matrix are then computed as

$$R_{pl}(\tau)=E\left\{\ddot{x}_p(t)\,\ddot{x}_l(t+\tau)\right\},\quad \text{and} \qquad \mathbf{T}_p=\mathbf{R}_p\cdot\mathbf{R}_p^T, \tag{9}$$

where $E\{\ \}$ indicates the expectation operator. The CoC matrix ($\mathbf{T}_p$) in Equation (9) is obtained by direct matrix operation from the covariance matrix $\mathbf{R}_p=\left[R_{pl_1}(t)\quad R_{pl_2}(t)\quad ...\quad R_{pl_m}(t)\right]^T$.

A new vector of condition index, $\mathbf{CCoC}$, can be formed from the components of the CoC matrix, $\mathbf{T}_p$ as

$$\mathbf{CCoC}=\left[(\mathbf{T}_p)_{1,1}\quad(\mathbf{T}_p)_{2,1}\quad(\mathbf{T}_p)_{2,1}\quad\cdots\quad(\mathbf{T}_p)_{m,1}\quad(\mathbf{T}_p)_{m,2}\quad\cdots\quad(\mathbf{T}_p)_{m,m}\right]^T \tag{10}$$

with the dimension of nm=($\frac{(1+m)\times m}{2}$ ×1). $(\mathbf{T}_p)_{i,j}$ is the element at i th row and j th column of the matrix $\mathbf{T}_p$.

4 PATTERN ASSURANCE CRITERION (PAC) METHOD

For health monitoring of in-service structures, the condition index vector $\mathbf{CCoC}$ will be monitored continuously. To detect if there is a change in the structural parameters, the CCoCs from different time periods are computed. The general measure is the modal assurance criterion. It is important to recognize that this least squares based form of linear regression analysis yields an indicator that is most sensitive to the largest difference between comparative values (minimizing the squared error) and results in a modal assurance criterion that is insensitive to small changes and/or small magnitudes [Randall & Allemang (2003)]. Therefore in this paper, a new technique is proposed for comparing the two vectors $\mathbf{CCoC}^{t_1}$ and $\mathbf{CCoC}^{t_2}$ as following.

The Pattern Assurance Criterion (PAC) value is defined as,

$$\text{If } \frac{\mathbf{CCoC}_i^{t_1}}{\mathbf{CCoC}_i^{t_2}} < 0,\ p_i = 0,\ \text{Else } p_i = \min\left(\frac{\mathbf{CCoC}_i^{t_1}}{\mathbf{CCoC}_i^{t_2}}, \frac{\mathbf{CCoC}_i^{t_2}}{\mathbf{CCoC}_i^{t_1}}\right),\ (i = 1:1:nm),\quad P = \sum_{i=1}^{nm} p_i \Big/ nm \tag{11}$$

where the subscript i denotes the i th element of the vectors $\mathbf{CCoC}^{t_1}$ and $\mathbf{CCoC}^{t_2}$, $\min(\bullet,\bullet)$ means choosing the smaller element from the two elements to ensure that an element less than unity is chosen as the ratio value. When $\mathbf{CCoC}^{t_1}$ and $\mathbf{CCoC}^{t_2}$ are identical, the PAC value P is 1.0, or else, P will get a value smaller than 1.0.

The PAC is a scalar formed as an algebraic average of the ratio of the corresponding elements from the two vectors in the comparison. This PAC will be shown later to be able to indicate small changes between two vectors. When the PAC value is below a given threshold, which can be defined from experience (e.g. the mean value from records of long-term health monitoring of a structure), the case may be considered to be possibly damaged. A trend analysis of the PAC values obtained from different time periods over the life span of the structure would reveal whether there is a damage occurrence in the structure. It is noted that this approach is independent of the structural model and it can be easily performed on experimental data from on-line health monitoring of structures.

5 EXPERIMENTAL ANALYSIS

5.1 Pearl River Huangpu Bridge and sensor array

The Pearl River Huangpu Bridge is located in Guangzhou, China. It is a cable-suspension bridge, consisting of a main span of approximately 1108 m, two side spans of 290 m and 350 m each. The roadway accommodates six lanes of traffic. The bridge was completed in 2008, and in 2009 the main span was instrumented with 22 accelerometers as part of a health monitoring project. Figure 1 shows the layout of the location of all the 22 sensors mounted on the bridge. A summary of the sensor numbering system and measurement directions which is adopted in the paper is presented in Table 1. Six accelerometers mounted in the vertical direction of the bridge deck are used for health monitoring of the main span of the bridge. The sampling frequency is 200 Hz. The acceleration responses from these six accelerometers are recorded continuously.

5.2 Covariance of Acceleration Responses

To obtain the structural modal parameters and compute the covariance of covariance, the covariance of acceleration responses needs to be computed firstly by Equation (9). To reduce the effect of non-white noise in the measurements, long-term data are used. In this paper, two hours' data, i.e. $2 \times 60 \times 60 \times 200 = 1440000$ time points are involved in Equation (9). Acceleration responses from Accelerometer #1 are chosen as reference. The computed $R_{1l}(\tau)$ with 15 second (3000 time points) are shown in Figure 2. It can be noticed that the covariance in Figure 2 (a) and (d) are similar. Other covariance curves do not have large values at the beginning. It is because the phases of acceleration responses at different locations are different. The vertical modal frequency of the bridge deck can be extracted from the covariance by Fourier transformation. The first 8 vertical modal frequencies are 0.215, 0.466, 0.881, 1.615, 2.097, 2.243, 2.855 and 3.337 Hz. This means that the largest cycle of the bridge is 4.65 s.

5.3 Covariance of Covariance of Acceleration Responses

From the computed covariance of acceleration, the covariance of covariance can be obtained from Equation (9). Since six acceleration responses are involved, the size of CoC matrix is 6×6. The reference point is still Accelerometer #1. The CoC matrices for different time period are computed and shown in Figure 3. The time periods are: 12:00 to 14:00, 14:00 to 16:00, 16:00 to 18:00, 18:00 to 20:00, 20:00 to 22:00, 22:00 to 24:00, April 12, 2010; 12:00 to 14:00, 14:00 to 16:00, 16:00 to 18:00, April 13, 2010. The data from other time periods in April 12, 2010 are bad data and they can not be used. The distributions for the CoC matrix are similar. For example the components at (1,1), (2,2) , (2,5), (3,3), (3,6) are bigger.

When the CoC matrix is converted into a vector, the condition index vector CCoC is obtained. To increase the number of components of CCoC, Accelerometer #1 to #6 are regarded as reference points, respectively. All six CoC matrices can be obtained in a similar way as described above. The diagonal components of CoC matrix are selected to form the CCoC with the size of 36×1. Since the magnitude of excitation is different at different time periods, which will affect the magnitude of CoC matrix, normalization is necessary. The normalization method is adopted that the Frobenius norm is 1, i.e $\frac{\mathbf{CCoC}}{\|\mathbf{CCoC}\|_2}$. The available data from 12:00, April 12, 2010 to 12:00, June 6, 2010 are analyzed. There are 46 sets of acceleration responses each lasting two hours which are analyzed and 46 CCoCs are formed and shown in Figure 4. It can be clearly seen that the distributions of CCoCs are similar. These distributions indicate that the bridge behaves regularly and no abnormity appears.

5.4 PAC Value Variation

To further estimate the condition of the bridge, the PAC value is computed by Equation (12) from the CoC matrix and the CCoC vector. The reference CCoC vector in Equation (11) is obtained from the mean value of all CCoCs from long-term measurements. All 46 PAC values are obtained and are shown in Figure 5. It can be seen that these PAC values are mostly in the range of 0.8 to 0.9. The smallest value is 0.7621 which is from the data measured during 10:00 to 12:00, April 15, 2010. The largest value is 0.9582 which is from 12:00 to 14:00, April 13, 2010. The fitted curve is also shown in the figure. The curve is steady. It means that the condition vectors CCoC are highly similar and the structural condition does not change obviously. Though the ideal PAC value for the unchanged structure is 1.0, the obtained results are acceptable because the bridge is subject to complex excitation and the excitation is possible to deviate from white noise distribution. If the bridge parameters change obviously, it is expected that the PAC value will decrease notably and the curve fitting will go downwards. It should be noted that further work should be done to validate that. But it is out of the scope of this paper.

6 CONCLUSIONS

The covariance of covariance of acceleration response of a structure is formulated and proved to be function of the modal frequencies and mode shapes limited by the sampling frequency. The experimental CoC matrix is easily obtained from acceleration responses. From the above experimental analysis of the measured data, it can be seen that from the variation of PAC values of CoC matrix, the condition of the bridge can be assessed. Only several accelerometers are used with continuous monitoring of acceleration responses. The CoC matrix is suitable for analyzing the huge amount of measurement data. The method relies on output-only data without analytical model. It can be said that the proposed CoC Matrix is appropriate for long-term health monitoring of in-service structures.

7 REFERENCES

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Table 1. Sensor locations

Category of sensor and station no.	**Sensor location**	**Sensor direction w.r.t. bridge**
#1	B-B section Truss downstream	z
#2	C-C section Truss downstream	z
#3	D-D section Truss downstream	z
#4	B-B section Truss top, i.e. upstream	z
#5	C-C section Truss top, i.e. upstream	z
#6	D-D section Truss top, i.e. upstream	z

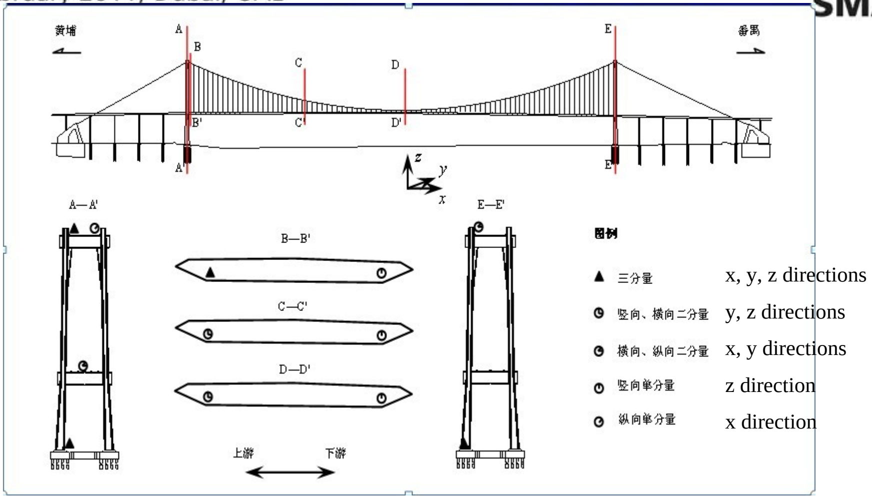

Figure 1 Accelerometer locations and directions for the instrumentation network on the Pearl River Huangpu Bridge

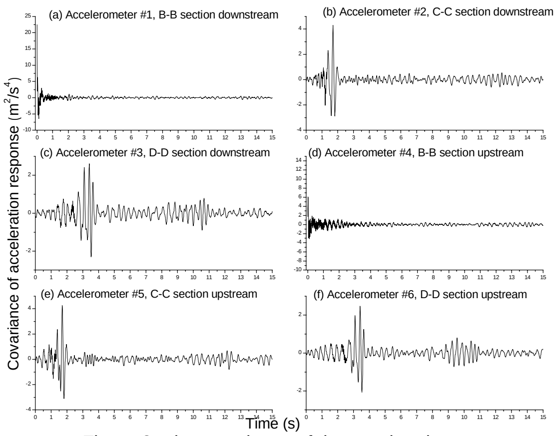

Figure 2 - the covariance of the acceleration responses with reference to Accelerometer #1

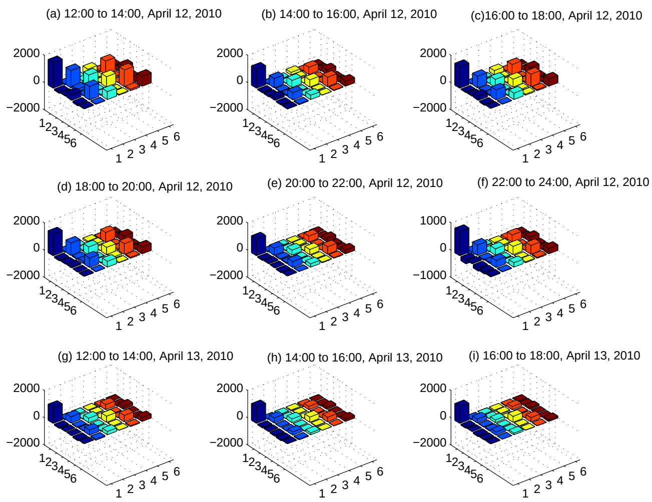

Figure 3 – the CoC matrix with reference of Accelerometer #1

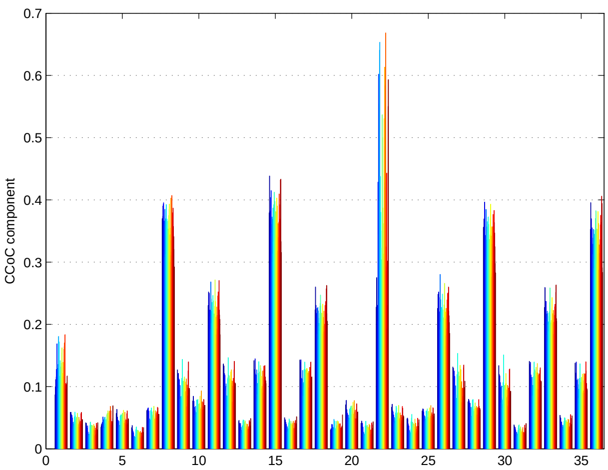

Figure 4 – the normalized condition vector CCoC

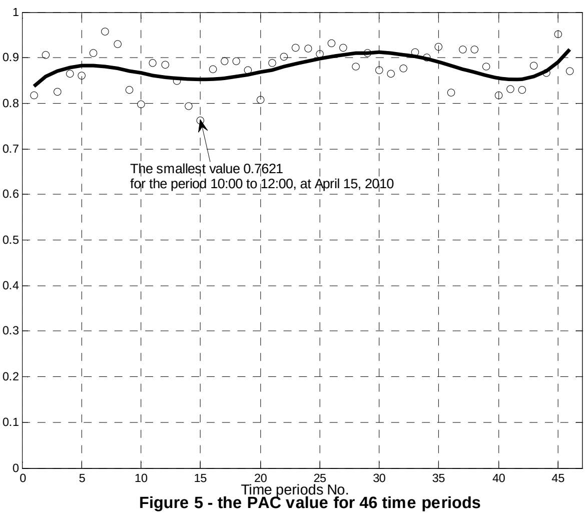

Figure 5 - the PAC value for 46 time periods