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Modeling of Bimodulus Materials with Applications to the Analysis of the Brazilian Disk Test

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ABSTRACT: This work presents a modeling approach to assess the dependence of the bimodularity of the elastic properties on the mechanical response of specimens under Brazilian test conditions. For a representative description of the statistical nature of the material, the correspondence between the tensile and compressive moduli is supposed to be fully described by a joint probability distribution. Conveniently, a Gaussian copula function is used to generate the distribution that correlates the bivariate elastic properties by specifying marginal univariate distributions estimated from experimental outcomes reported in the literature. Monte Carlo simulations of disk-shaped specimens under diametrical compression using randomly selected values of the elastic modulus as input into a finite element program are conducted under identical loading and boundary conditions. The constitutive law is based on the plane-stress assumption. As an illustrative case, the proposed approach is used to provide insight into the effects of bivariate random parameters on the mechanical behavior of a white marble. Numerical results are compared to a previously developed analytical solution framed on the same modulus theory and to a deterministic numerical response based on mean values. The comparison between the numerical and theoretical analyses proved that the proposed stochastic scheme can effectively characterize the spatial variability of the mechanical behavior of bimodular materials.

1 INTRODUCTION

Some brittle materials used in structural engineering, such as concrete and rocks, exhibit different mechanical properties under tensile or compressive stresses. These properties show dispersion in their values not only due to the variability of the natural processes of material formation, but also to factors such as (1) the accuracy of the methods used for their characterization; (2) the limited number of experimental observations; and (3) the inaccuracy of the statistics considered. Several studies have formulated models predicting linear elastic relationships between stresses and deformations with deterministic and independent properties of the directionality of internal stresses. However, it is prudent to consider that these properties are stochastic variables with magnitudes that could be unequal to traction to compression [Haimson et al. (1974)]. After its introduction by Carneiro (1943), the diametral compression test or better known as the Brazilian test has facilitated the evaluation of tensile mechanical properties of fragile materials. In this test, disc-shaped specimens are loaded by applying a pair of diametrical forces, inducing a flat tensional state with approximately constant tensile stresses along the load application plane. Due to their increasing acceptance, numerous studies have been formulated to experimentally validate elastic properties in rocks [Patel et al. (2018)] or to evaluate the anisotropic behavior of stratified sandy rocks [Khanlari et al. (2015)].

The main focused of this study was to propose a stochastic modeling of the mechanical behavior of materials that exhibit bimodular elastic properties. As a first approximation, the model sought to predict the variability of stress states due to fluctuations derived from the nature of the material. In our case, the influence of the uncertainties of the bi-modularity of the elastic properties of a fine-grained white marble was considered. For this purpose, a Gaussian copula was proposed to statistically characterize the random dependence of the moduli and the resolution of the constitutive equations is achieved using the finite element method in combination with a Monte Carlo sampling of the copula considered.

2 PROBLEM FORMULATION

For the purpose of this study, the material was assumed as continuous and homogenous. Thus, the relationship between stress and strain is presented using the classical format:

$$\sigma_{ij} = 2G\varepsilon_{ij} + (K - \frac{2}{3}G)e\delta_{ij} \tag{1}$$

where σ_{ij} and ε_{ij} represent the stress and strain tensors, respectively; e is the first invariant of the strains (i.e., $e = \varepsilon_{ii}$); K and G are the volumetric and the shear modulus, respectively. K and G can be written in terms of the elastic modulus E and the Poisson's ratio ν as:

$$K = \frac{E}{3(1-2\nu)}; \qquad G = \frac{E}{2(1+\nu)} \tag{2}$$

If the mechanical behavior of the material along any direction is different in tension or compression, then, K is assumed to be dependent of the local component of hydrostatic strain; thus:

$$K = K(e) = \frac{E(e)}{3(1-2\nu)} \tag{3}$$

where the Poisson's ratio is isotropic, then, E can be expressed as a function of the tensile and compressive moduli. E_t and E_c are expressed in the following manner:

$$E = \begin{cases} E_t & \text{if } e \geq 0 \\ E_c & \text{if } e < 0 \end{cases} \tag{4}$$

For the considered case, the shear modulus is assumed as independent of the state of deformation. The following section describes the approximation employed to estimate the joint distribution of E_t and E_c using the copula approach.

3 MODULUS ESTIMATION USING THE COPULA APPROACH

The copula approach is powerful tool used to model dependence. Consider that the elastic properties of the material exhibit a bivariate random nature and that the copula theory appropriately describes the correlation structure by means of a relational function between the marginal distributions and the joint distribution of the variables [Nelsen (2006)]. Suppose that U and V are random variables with distribution functions G and H, respectively. In this way, (U, V) is defined as a random vector of *n* pairs of moduli. Then, let us think that the pairs (u, v) represent predicted simultaneous values in the ranges of U and V with a joint distribution function $F(u, v)$. Thus, $G(u)$ and $H(v)$ constitute the one-dimensional functions of marginal distributions which are obtained from the integration of $F(u, v)$ on the values of U and V. That is to say:

$$G(u) = \int_{-\infty}^{\infty} F(u,v)du \qquad \text{and} \qquad H(v) = \int_{-\infty}^{\infty} F(u,v)dv \tag{5}$$

These marginal functions are non-decreasing and describe uniform standard distributions. Thus, $G, H\colon \mathrm{R} \rightarrow \mathrm{I} := [0,1]$. Now, let us postulate the existence of a copula function C such that $F(u,v) = C(G(u), H(v))$, where $C\colon \mathrm{I}^2 \rightarrow \mathrm{I}$. If the inverse functions of Equation (5) are given as $u = G^{(-1)}(x)$ and $v = H^{(-1)}(y)$. Then, it can be proven that $C(x,y) = F(G^{(-1)}(x), H^{(-1)}(y))$. Thus, the density, joint distribution and copula are defined as:

$$f(u,v) = \frac{\partial^2 F(u,v)}{\partial u \partial v} \quad \text{and} \quad c(x,y) = \frac{\partial^2 C(x,y)}{\partial x \partial y} \tag{6}$$

where the joint density is reduced to:

$$f(u,v) = c\big(G(u), H(v)\big)\, g(u)h(v) \tag{7}$$

and the conditioned distribution is expresses as:

$$F_{U|V}(v) = \tilde{C}(G(u), H(v)) \tag{8}$$

For example, if the dependence relation is symmetrical and elliptical, such and in the case of a bivariant Gaussian distribution, and it is considered the existence of a correlation coefficient $\rho(U,V)$, the copula has the following form:

$$C(x,y,\rho) = \Theta\big(\Phi^{-1}(x), \Phi^{-1}(y)\big) \tag{9}$$

where Φ is the normal standard distribution expressed as follows:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{10}$$

and Θ is the bivariant normal standard distribution [Joe (2015)]. Thus, the copula's normal distribution is given in the following manner:

$$C(u,v,\rho) = \frac{1}{\sqrt{1-\rho^2}} \exp\left(\frac{u^2+v^2-2\rho uv}{2(1-\rho^2)} + \frac{u^2+v^2}{2}\right) \tag{11}$$

where $u = \Phi^{-1}(x)$ and $v = \Phi^{-1}(y)$. The following expression is used when the moduli followed a log-normal distribution:

$$G(u|\mu,\sigma) = \frac{1}{\sigma u \sqrt{2\pi}} \exp\left(-\frac{(\ln u - \mu)^2}{2\sigma^2}\right) \tag{12}$$

Then, the corresponding marginal density functions are as those shown in the following section.

4 NUMERICAL EXAMPLE AND DISCUSSION

This section introduces the numerical model of the behavior of fine-grained white marble specimens tested under diametrical compression. To simulate this test, the disc-shaped specimens are subjected to a diametrical compression load applied by a pair of curved supports as established by the International Society of Rock Mechanics [Franklin (1985)]. The test setup is shown in Figure 1(a). The disc has a diameter of 50.0 mm and thickness of 25.0 mm. The steel loading plates have a curvature of 37.5 mm. The numerical resolution was performed using the finite element method (FEM). Considering that the marble object to this study is homogenous, the symmetry of the geometry and the load is used to impose appropriate boundary conditions and to reduce the size of the domain to a quarter of the total geometry as shown in Figures 1(b) and 1(c). As seen in Figure 1(c), the specimen disk and steel plates are discretized using 479 and 55 triangular elements, respectively. A total displacement of 0.04 mm was applied to the specimens under a static load condition with increments of 0.5 %. Due to symmetry, a contact condition was assumed without an initial preload and without friction between the specimen and the curved support.

Jianhong et al. (2009) conducted an experimental evaluation of bilinear elastic white marble with mean tensile and compressive modulus values of 66.80 and 76.86 MPa, respectively. For the simulations, it was estimated that the corresponding standard deviation and were 11.16 MPa and 14.67 MPa with a correlation coefficient of 1.15. The Poisson's ratio was assumed as isotropic with a deterministic value of 0.33. The plates are made of steel with an elastic modulus of 200 GPa and a Poisson's ratio of 0.30. A total of 1000 simulations were performed using the same number of realizations (observations) of pairs of moduli as input parameters of the model. The marginal distributions were adjusted using a one-dimensional log-normal model. The dependence between the variables was modeled using the Gaussian elliptic copula described in the previous section using Monte Carlo sampling to cover a wide range of possible scenarios contained in the joint distribution. Figure 2(a) shows the marginal density functions for the tensile and compressive moduli, estimated from the corresponding average values and standard deviations. Figure 2(b) shows the embodiments of the joint values of tension and compression moduli whose dependence is expressed in terms of the copula. The positive dependence of the moduli and their variability around the corresponding average values is shown in Figure 2(b). For the pairs of moduli shown on this graph, the linear regression (red line) corresponds to the expression that follows:

$$E_t = 1.07\ (1.032{\sim}1.107)E_c - 15.30(-18.17{\sim}-12.42) \qquad (13)$$

The values in parentheses indicate the coefficients for an confidence interval of 95%.

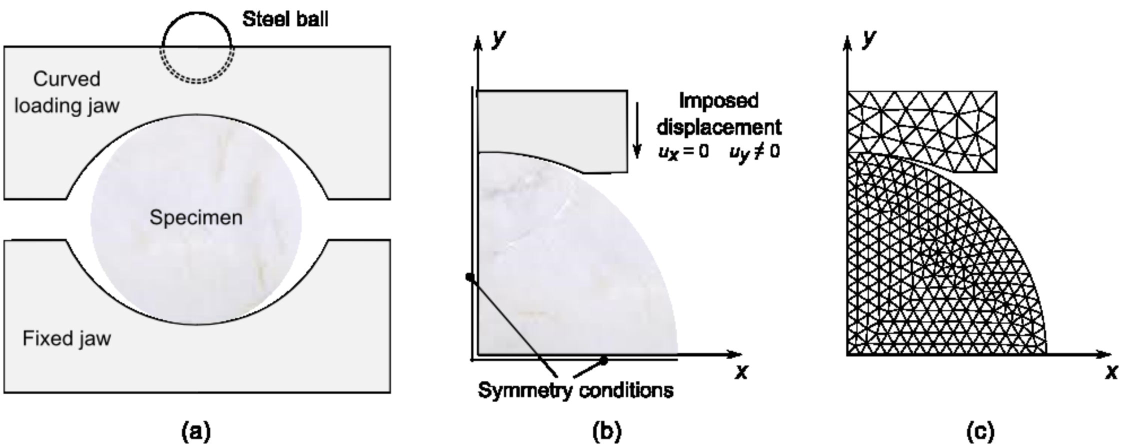

Figure 1. Brazilian test setup. (a) Test specimen; (b) simplified model; (c) finite element mesh.

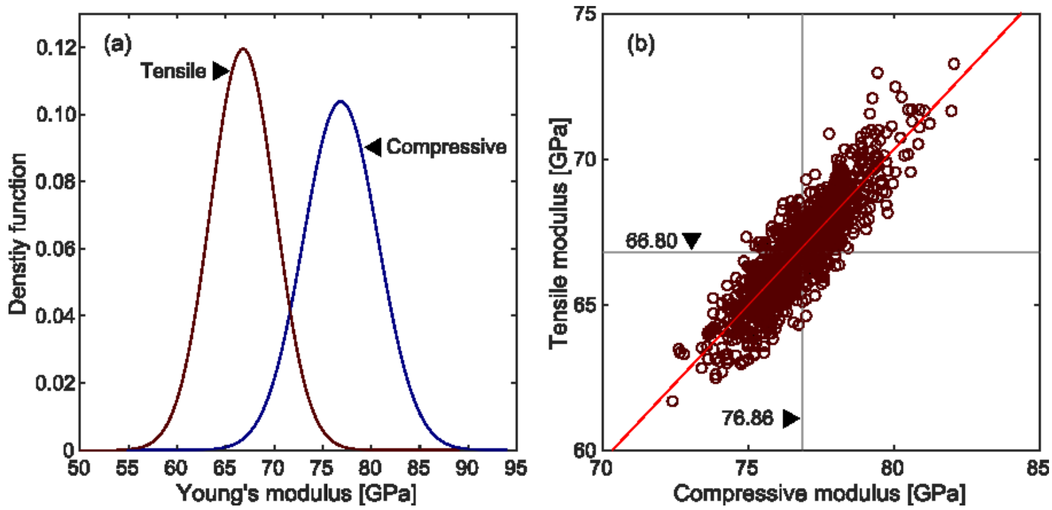

Figure 2. Gaussian copula. (a) Marginal density functions of the tensile and compressive moduli; (b) bivariant sample of 1000 points of a Gaussian copula.

A statistical sample of the mechanical response of the material was obtained from the simulation results. Given the fact that the Brazilian test is a test used mainly to indirectly obtain the tensile strength, the variable considered to analyze the uncertainty of the response is the normal stress along the x direction. For comparative purposes, Figure 3 presents the stress distribution of the material for three case studies (two were deterministic cases and one was a stochastic case).

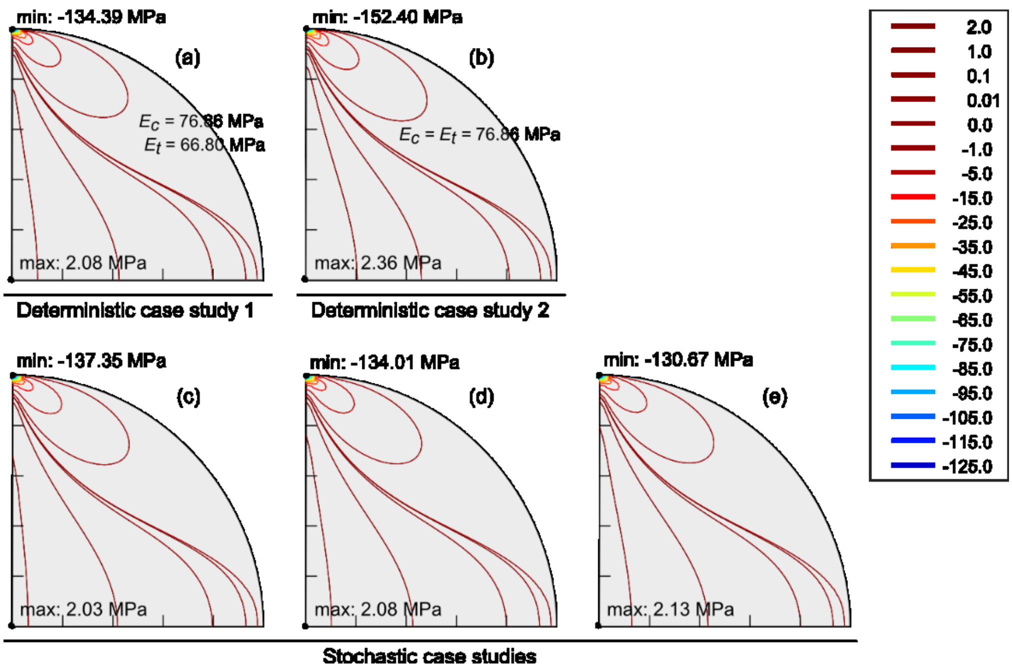

Figure 3. Stress distribution along the x axis of study cases. Normal stress values are expressed in MPa.

The stress distribution of Figure 3(a) corresponds to the first deterministic case, where tensile and compressive moduli equal to the experimental average values are admitted. For this case, the maximum tensile and compressive stresses have values of 2.08 and -134.39 MPa, respectively, and are located at the center of the disk (point of application of the load). Figure 3(b) concerns a second deterministic case where it is assumed that the modulus is independent of the load direction and has a magnitude equal to the average value of the experimental compression modulus. For this case, the maximum tensile and compressive stresses were 2.6 and -152.40 MPa, respectively. From Figure 3(a) and 3(b), it is noted the existence of a relationship between the increase of tensile stiffness and the increase of stress in the material. An increment of 15% of the tensile modulus with respect to the first deterministic case constituted an increase of approximately 13% of the stresses along the vertical axis of the specimen. Figures 3(c) – 3(e) correspond to the predicted stress distributions for the third case study (the stochastic case). Figure 3(d) shows the distribution of the average stresses, while Figure 3(e) illustrate te distributions corresponding to the interval plus/minus the standard deviation. Regarding the average values, it was observed a ± 2.5% variation of the maximum stresses. Note that unlike the two deterministic cases, in the stochastic case it was observed that as the maximum tensile stresses increased, the maximum compressive stresses exhibited a decrease proportional to that observed under tension.

Figure 4 shows the response surface for the maximum stress that was obtained from the stochastic analysis of the test. This response surface describes the relationship between the independent variables (tensile and compression moduli) and the maximum tensile stress along the x direction. In terms of the moduli, the response surface was approximated as a first-order of the form $\sigma_x = \alpha + \beta E_c + \gamma E_t$. By using the least-square method, we have:

$$\sigma_x = 9.964(10^{-15}) + 0.002721 E_c + 0.02802 E_t \tag{14}$$

for and confidence interval equal to 95%, the parameters were bounded to the following ranges:

$$\begin{gathered} -0.0001029 \leq \alpha \leq 0.0003022 \\ 0.002716 \leq \beta \leq 0.002726 \\ 0.02801 \leq \gamma \leq 0.02802 \end{gathered} \tag{15}$$

Due to the characteristics of the response surface, it is evident that the stresses are much more sensitive to changes in the tensile modulus than to variations in the compression modulus.

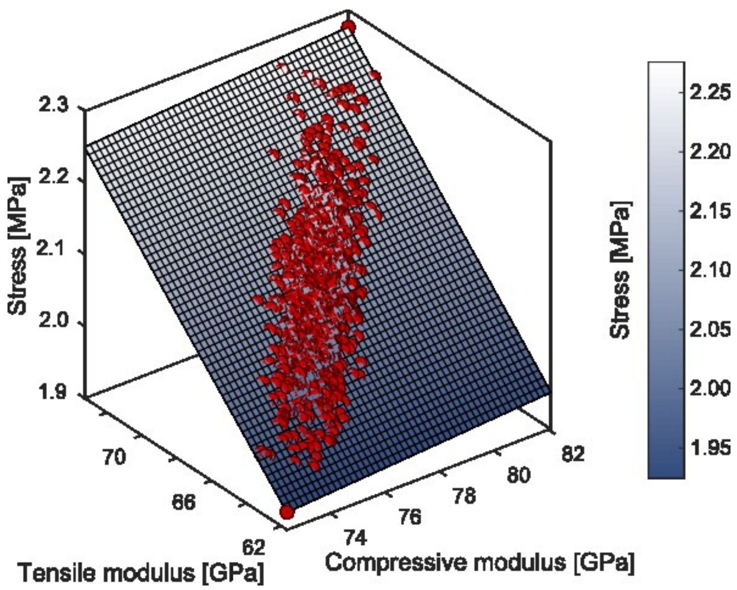

Figure 4. Response surface of the maximum tensile stress. Note: The red circles represent data obtained for the stochastic case.

5 CONCLUDING REMARKS

An experimental stochastic model was proposed to study the mechanical behavior of bimodular materials. The elastic properties dependence was characterized by means of the copula approach. A study case of a white marble tested under diametrical compression was presented. For the proposed model, the material was described as isotropic and homogenous while the randomness and correspondence of the tensile and compressive elastic properties was characterized by a joint probability distribution given a Gaussian copula. The resolution of the constitutive equations was conducted by using the finite element method in combination with the Monte Carlo copula sampling technique. The simulations were intended to capture the randomness influence of the modulus on the tensile stress response and compare the results with two deterministic cases. The methodology can be extended to marginal distributions with copulas that capture the correspondence among multidimensional variables such as (1) compressive and tensile moduli; (2) tensile and compressive strengths; (3) heterogenous elastic properties expressed in terms of stochastic fields or response surfaces of nonlinear constitutive material laws.

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